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# D-branes on orbifolds with discrete torsion and topological obstruction

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Abstract: We find the orbifold analog of the topological relation recently found by Freed and Witten which restricts the allowed D-brane configurations of type-II vacua with a topologically non-trivial flat $B$-field. The result relies in Douglas proposal — which we derive from worldsheet consistency conditions — of embedding projective representations on open string Chan-Paton factors when considering orbifolds with discrete torsion. The orbifold action on open strings gives a natural definition of the algebraic K-theory group — using twisted cross products — responsible for measuring Ramond-Ramond charges in orbifolds with discrete torsion. We show that the correspondence between fractional branes and Ramond-Ramond fields follows in an interesting fashion from the way that discrete torsion is implemented on open and closed strings.

Keywords: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

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## Contents





## 1. Introduction, results and conclusions

Orbifolds [1] in string theory provide a tractable arena where CFT can be used to describe perturbative vacua. In the more conventional geometric compactifications, geometry and topology provide powerful techniques in describing the long wavelength approximation to string theory. In a sense, geometric compactifications and orbifolds provide a sort of dual description. In the latter, CFT techniques are available but topology is less manifest. On the other hand, conventional Calabi-Yau compactification [2] lacks an exact CFT formulation but a rich mathematical apparatus aids the analysis of the corresponding supergravity approximation.

Perhaps surprisingly, orbifold CFT [1, 3] seems to be able to realize topological relations satisfied by geometric compactifications from worldsheet consistency conditions. A nice example of this phenomenon is the interpretation [4] of a restriction imposed by modular invariance as the analog of the topological constraint requiring the space-time manifold to have a vanishing second Stieffel-Whitney class.

The original motivation of this work was to find the analog of the topological formula recently found by Freed and Witten[1] [7] in the context of orbifolds. Let us briefly explain their results in a language that will be convenient for what follows. Given a space-time manifold $X$ and a submanifold $Y \subset X$, their formula constraints the configuration of D-branes allowed to wrap $Y$. A careful analysis [7] of string worldsheet global anomalies in the presence of D-branes and a flat Neveu-Schwarz B-field — so that the curvature $H = dB$ is zero — imposes, for a class of backgrounds, the following topological relation[2]

$$i^*[H] = 0\,, \tag{1.1}$$

[1] Several aspects of this topological relation had already been considered by Witten in [5, 6].

[2] This formula can have an additional term that depends on the topology of $Y$. We will ignore this correction since it vanishes when the second Stieffel-Whitney class of $Y$ is trivial and the orbifold model satisfies a relation which can be identified with the vanishing of this class.

where $[H] \in H^3(X, \mathbb{Z})$ determines the topological class of the B-field and $i^*[H]$ is the restriction of $[H]$ to the D-brane worldvolume $Y \subset X$. Since $[i^*[H]] \in H^3(Y, \mathbb{Z})$ is a torsion class in cohomology, so that there is a smallest non-zero integer $m$ such that $m \cdot [i^*[H]] \simeq 0$, the anomaly relation (1.1) can only be satisfied whenever there are a multiple of $m$ D-branes wrapping $Y \subset X$. Therefore, in a background with a topologically non-trivial flat B-field such that $m \cdot [i^*[H]] \simeq 0$, the charge of the minimal D-brane configuration wrapping $Y$ is $m$ times bigger than in a background with trivial B-field. The anomaly relation (1.1) expresses in topological terms that D-branes in string theory are not pure geometric constructs, but that the allowed configurations may depend on discrete choices — like the choice of $[H]$ — of the string background. In some cases, this fact is realized by the K-theory classification [8, 6] of Ramond-Ramond charges.[3] For example, whenever $[H]$ is not trivial, type-IIB D-brane charges are classified by the twisted topological K-theory group $\mathrm{K}_{[H]}(X)$ [6, 13] instead of the more conventional group $\mathrm{K}(X)$ used whenever $[H]$ is trivial.

In this note we find the analogous phenomenon for orbifold models. Given a compact orbifold $\mathbb{T}^6/\Gamma$ with discrete torsion[4] [4], we show that the charge of the minimal D6-brane configuration wrapping the orbifold is an integer multiple bigger than the minimal charge when one considers conventional orbifolds (without discrete torsion). This result parallels the consequences that stem from (1.1). Roughly speaking, turning on discrete torsion in the orbifold corresponds to turning on a flat topologically non-trivial $B$-field in a geometric compactification and the conventional orbifold corresponds to the case where the $B$-field is trivial. This suggest that discrete torsion in string theory is intimately related to torsion in homology [15]–[19].[5]

A crucial ingredient in deriving this result is a careful treatment of open strings in orbifolds with discrete torsion. Douglas [20] has proposed that discrete torsion should be implemented on open strings by embedding a projective representation of the orbifold group on Chan-Paton factors. Whether $\Gamma$ admits discrete torsion and projective representations depends on its cohomology via $H^2(\Gamma, \mathrm{U}(1))$. This alone suggest the correlation between discrete torsion and projective representations. We show that worldsheet consistency conditions uniquely determine the action of the orbifold group on Chan-Paton factors once a closed string orbifold model is specified. This result can be derived by demanding that open and closed strings

[3] The work of Sen describing D-branes via tachyon condensation of unstable systems — see for example [9, 10, 11] — was crucial in making the identification between D-branes and K-theory. See [12] for a type-IIA discussion.

[4] The simplest supersymmetric orbifold model with discrete torsion appears when the orbifold is three complex dimensional. For concreteness, we will study in section 3 the case $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_{n'}$. In [14], the values of $n$ and $n'$ so that $\Gamma$ acts cristallographically were found.

[5] In many examples the relation is not direct and the correspondence between torsion in homology and discrete torsion is only visible after an irrelevant perturbation, as in for example [17].

interact properly in the orbifold, so that the orbifold group $\Gamma$ is conserved by their interactions. A careful account of what is discrete torsion is essential in this derivation and we shall present its description in section 2.

The effects of discrete torsion can be incorporated to define a K-theory group which measures Ramond-Ramond charges in orbifolds with discrete torsion. In the algebraic[6] approach to equivariant K-theory via cross products [22], one incorporates discrete torsion by twisting the cross product by a cocycle[7] $c \in H^2(\Gamma, \mathrm{U}(1))$ corresponding to the choice of discrete torsion and projective representation. It turns out that whenever the action of a finite group $\Gamma$ on $X$ is free, the algebraic K-theory of twisted cross products $\mathrm{K}_\Gamma^{[c]}(X)$ is isomorphic [23] to the twisted topological K-theory group $\mathrm{K}_{[H]}(X/\Gamma)$ which classifies D-branes in a background with topologically non-trivial flat B-field. The use of projective representations — and therefore of cocycles — provides a definition of K-theory which is the orbifold generalization of $\mathrm{K}_{[H]}(X)$.

We show that the minimal D-brane charge for six-branes wrapping $\mathbb{T}^6/\Gamma$ is larger for orbifolds with discrete torsion than for conventional orbifolds by explicitly computing the D-brane charge. The D6-brane charge can be extracted from a disk amplitude with an insertion of the corresponding untwisted Ramond-Ramond vertex operator. The same result can be obtained as in [24] using the boundary state formalism [25]. As shown in [26, 24], the properly normalized untwisted Ramond-Ramond six-brane charge is given by

$$Q = \frac{d_R}{|\Gamma|}, \tag{1.2}$$

where $d_R$ is the dimension of the $R$-representation[8] of $\Gamma$ acting on the Chan-Paton factors and $|\Gamma|$ is the order of the group. The minimal charge is therefore obtained by taking the smallest irreducible representation of $\Gamma$. For open strings in conventional orbifolds one must use standard (vectorial) representations of $\Gamma$. For any discrete group $\Gamma$, the smallest irreducible vectorial representation is always one-dimensional.[9] As shown in section 2, particular projective representations must be used when dealing with orbifolds with discrete torsion. A simple and important property of projective representations is that there are no non-trivial[10] one-dimensional projective

[6]See [21] for prior use of the algebraic approach to K-theory in string theory.

[7]The multiplication law of the cross product $C(X) \propto \Gamma$, where $C(X)$ is the algebra of continous functions on $X$, is twisted by a cocycle $c \in H^2(\Gamma, \mathrm{U}(1))$ and defines and associative group ring generated by the elements of $\Gamma$ with $C(X)$ coeffitients. The Grothendieck group of the twisted crossed product yields the K-theory group $\mathrm{K}_\Gamma^{[c]}(X)$. See [22] for a more complete discussion.

[8]In the more conventional setup of branes transverse to a non-compact orbifold, a bulk brane is described by the regular representation so that $Q = 1$ and a brane stuck at the singularity by an irreducible representation and carries fractional untwisted charge [26].

[9]One always has the trivial representation where each element $g_i \in \Gamma$ is representated by 1.

[10]Any group $\Gamma$ can have projective representations. The important issue is whether a given projective representation can be redefined to become a vectorial one. This will become more clear in section 2.

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representations. Therefore, given a model with orbifold group $\Gamma$ and non-trivial $H^2(\Gamma, \mathrm{U}(1))$, the charge of the minimal D6-brane configuration when discrete torsion is turned on is given in terms of the charge of the minimal D6-brane configuration when discrete torsion is turned off by

$$Q_{\text{dis. tors.}} = d_R^{\text{proj}} Q_{\text{convent.}} \,. \tag{1.3}$$

$d_R^{\text{proj}} > 1$ is dimension of the smallest irreducible projective representation of $\Gamma$. Thus, the charge of a D6-brane wrapping the entire orbifold is always larger when one consider orbifolds with discrete torsion than when one considers conventional orbifolds.

The correlation between discrete torsion in the closed string sector and the use of projective representations on open strings provides a natural description of fractional branes [27, 26] in these models. For simplicity, let's consider D0-branes sitting at a point in a conventional non-compact orbifold $C^3/\Gamma$. The charge vector of any zero-brane state lies in a charge lattice generated by a basis of charge vectors. Each irreducible representation of $\Gamma$ is associated with a basis vector of the charge lattice. This result can be shown both from CFT [26, 24] and the K-theory approach [8, 6] to D-brane charges using equivariant K-theory [6, 28, 29]. The closed string spectrum yields a massless Ramond-Ramond one-form potential for each twisted sector, but there are as many twisted sectors as irreducible representations of $\Gamma$, so that indeed one can associate a generator of the charge lattice with each irreducible representation. A particular zero-brane state is uniquely specified by the choice of representation of $\Gamma$ on its Chan-Paton factors, but any representation of $\Gamma$ can be uniquely decomposed into a particular sum of its irreducible ones. Therefore, the states associated with the irreducible representations can be used as a basis of zero-brane states. This is realized by equivariant K-theory since $\mathrm{K}_\Gamma(C^3) \simeq R(\Gamma)$, where $R(\Gamma)$ is the representation ring of $\Gamma$. The CFT argument goes through in the presence of discrete torsion. That is, any zero-brane state has a unique decomposition in terms of the states associated with the irreducible projective representations. The question is if there are as many massless Ramond-Ramond one-form fields as irreducible projective representations, so that one can associate a generator of the charge lattice to each irreducible representation. This a priori seems non-trivial since the projection in the twisted sector is different[11] in the presence of discrete torsion and generically the projection removes these massless fields. The matching between massless Ramond-Ramond fields and irreducible projective representations follows in an interesting way from the algebraic properties of discrete torsion. It turns out that the number of irreducible projective representations of $\Gamma$ equals the number of $c$-regular[12] elements of $\Gamma$ [30]. Moreover, the closed string spectrum in the

[11] See section 2 for more details.

[12] A group element $g_i \in \Gamma$ is $c$-regular if $c(g_i, g_j) = c(g_j, g_i)\ \forall g_j \in \Gamma$(we take $\Gamma$ abelian for simplic-

twisted sectors associated with $c$-regular elements is identical[10] to the corresponding twisted sector spectrum in the conventional orbifold (without discrete torsion) which do have massless Ramond-Ramond fields. Thus, each irreducible projective representation is associated with a generator in the charge lattice even when there is non-trivial discrete torsion. This intuitive result, which follows from algebraic properties of cocycles, ties in a nice way the effects of discrete torsion on open and closed strings. From this CFT result, one is naturally led to conjecture that the K-theory of twisted cross products $K_\Gamma^{[c]}(C^3) \simeq R_{[c]}(\Gamma)$, where now $R_{[c]}(\Gamma)$ denotes the module of projective representations of $\Gamma$ with cocycle $c$.

The organization of the rest of the paper is the following. In section 2 we explain the inclusion of discrete torsion in closed string orbifolds and relate its properties to the topology of the orbifold group $\Gamma$. We analyze D-branes in these orbifolds and derive from worldsheet consistency conditions the necessity to use projective representations when analyzing open strings in orbifolds with discrete torsion. In section 3 we find the orbifold analog of the result by Freed and Witten [7], present several examples and describe the charges of fractional branes in orbifolds with discrete torsion.



## 2. Open and closed strings in orbifolds with discrete torsion

The dynamics of a D-brane at an orbifold singularity provides a simple example of how the geometry of space-time is encoded in the D-brane worldvolume theory (for a partial list of references see [31]–[41] and for interesting applications to AdS/CFT see [42, 43] ). The low energy gauge theory on the brane is found by quantizing both open and closed strings on the orbifold [31]. Closed string modes appear as parameters in the gauge theory such as in Fayet-Iliopoulos terms and in the superpotential. Open string modes provide gauge fields and scalars which describe the fluctuations of the brane. In this section we will show that consistency of interactions between open and closed strings require embedding an appropriate projective representation of the orbifold group on the open string Chan-Paton factors when studying orbifolds with discrete torsion.[13] We will start by briefly explaining the essentials of discrete torsion and describing the closed string spectrum of these models. This will be crucial in determining the appropriate projection on open strings.

The spectrum of closed strings on an orbifold $X/\Gamma$ — with abelian $\Gamma$ — is found by quantizing strings that are closed up to the action of $\Gamma$ and projecting onto $\Gamma$

ity), where $c$ is a cocycle determining the projective representations $\gamma(g_i)\gamma(g_j) = c(g_i, g_j)\gamma(g_i g_j)$ and the discrete torsion phase $\epsilon(g_i, g_j) = c(g_i, g_j)/c(g_j, g_i)$. $\epsilon = 1$ for $c$-regular elements. See sections 2 and 3 for more details.

[13] Recently, [44, 45, 19] have considered the gauge theory on branes on an orbifold with discrete torsion.

invariant states. When $\Gamma$ is an abelian group,[14] one must quantize and project $|\Gamma|$ closed strings. This is reflected in the partition function of the orbifold by it having $|\Gamma|^2$ terms corresponding to all the possible twists along the $\sigma$ and $\tau$ directions of the worldsheet. One loop modular invariance allows each term in the partition function to be multiplied by a phase

$$Z = \sum_{g_i, g_j \in \Gamma} \epsilon(g_i, g_j) Z_{(g_i, g_j)}\,, \tag{2.1}$$

such that $\epsilon(g_i, g_j)$ is invariant under an $\mathrm{SL}(2,\mathbb{Z})$ transformation[15] and $Z_{(g_i,g_j)}$ is the partition function of a string closed up to the action of $g_i \in \Gamma$ with an insertion of action of $g_j \in \Gamma$ in the trace.

As first noted by Vafa [4], modular invariance on higher genus Rieman surfaces together with factorization of loop amplitudes imposes very severe restrictions on the allowed phases. Orbifolds models admitting these non-trivial phases are usually referred as orbifolds with discrete torsion. As we shall briefly explain in a moment, whether a particular orbifold model admits such a generalization depends on the topology of the discrete group $\Gamma$.

In [4], Vafa showed that $\epsilon$ must furnish a one dimensional representation[16] of $\Gamma$

$$\epsilon(g_i, g_j g_k) = \epsilon(g_i, g_j)\epsilon(g_i, g_k) \tag{2.2}$$

for each $g_i \in \Gamma$. This provides a natural way to take $\epsilon$ into account when computing the closed string spectrum. In orbifolds with discrete torsion, the spectrum in the $g_i$ twisted sector is obtained by keeping those states $|s\rangle_i$ in the single string Hilbert space that satisfy

$$g_j \cdot |s\rangle_i = \epsilon(g_i, g_j)|s\rangle_i \qquad \forall g_j \in \Gamma \tag{2.3}$$

States satisfying (2.3) transform in a one dimensional representation of $\Gamma$. In this language, the spectrum of conventional orbifolds transform in the trivial one-dimensional representation of $\Gamma$ where $\epsilon \equiv 1$.

Discrete torsion is intimately connected with the topology of $\Gamma$ via [4]

$$\epsilon(g_i, g_j) = \frac{c(g_i, g_j)}{c(g_j, g_i)}. \tag{2.4}$$

Here $c \in \mathrm{U}(1)$ is a two-cocycle, which is a collection of $|\Gamma|^2$ phases satisfying the following $|\Gamma|^3$ relations

$$c(g_i, g_j g_k)c(g_j, g_k) = c(g_i g_j, g_k)c(g_i, g_j), \qquad \forall g_i, g_j, g_k \in \Gamma. \tag{2.5}$$

[14] In this paper we shall consider $\Gamma$ abelian only. It is straightforward to generalize to non-abelian groups.

[15] That is $\epsilon(g_i, g_j) = \epsilon(g_i^a g_j^b, g_i^c g_j^d)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

[16] For non-abelian $\Gamma$, $\epsilon(g_i, g_j)$ must be a one dimensional representation of the stabilizer subgroup $N_{g_i} \in \Gamma$, where $N_{g_i} = \{g_j \in \Gamma, g_i g_j = g_j g_i\}$.

The set of cocycles can be split into conjugacy classes via the following equivalence relation compatible with (2.5)

$$c'(g_i, g_j) = \frac{c_i c_j}{c_{ij}} c(g_i, g_j). \tag{2.6}$$

One can show from the definition of discrete torsion in (2.4) that indeed $\epsilon$ is a one dimensional representation[17] of $\Gamma$. Moreover, the discrete torsion phase (2.4) is the same for cocycles in the same conjugacy class. Therefore, the number of different orbifold models that one can construct is given by the number of conjugacy classes of cocycles. Topologically, equivalence classes of cocycles of $\Gamma$ are determined by its second cohomology[18] group $H^2(\Gamma, \mathrm{U}(1))$. Summarizing, given a discrete group $\Gamma$ there are as many possibly different orbifold models that one can construct as the number of elements in $H^2(\Gamma, \mathrm{U}(1))$.

Placing D-branes in these vacua requires analyzing both open and closed strings in the orbifold. The closed string spectrum was summarized in the last few paragraphs in a language that will be convenient when considering open strings. The most general action on an open string is obtained by letting $\Gamma$ act both on the interior on the string (the oscillators) and its end-points (the Chan-Paton factors). The open string spectrum is found by keeping all those states invariant under the combined action of $\Gamma$ [46, 31]

$$|s, ab\rangle = \gamma(g_i)^{-1}_{aa'} |g_i \cdot s, a'b'\rangle \gamma(g_i)_{b'b}\,, \tag{2.7}$$

where $s$ is an oscillator state and $ab$ is a Chan-Paton state. Consistent action on the open string state and completeness of Chan-Paton wavefunctions demand $\Gamma$ to be embedded on Chan-Paton factors by matrices the represent $\Gamma$ up to a phase

$$\gamma(g_i)\gamma(g_j) \propto \gamma(g_i g_j)\,. \tag{2.8}$$

This seems lo leave some arbitrariness since $\Gamma$ may have several classes of representations. As we will show shortly the arbitrariness is removed once closed strings are also taken into account. In particular $\Gamma$ may have several classes of projective representations where group multiplication is realized only up to a phase. The most general such representation is given by

$$\gamma(g_i)\gamma(g_j) = c(g_i, g_j)\gamma(g_i g_j)\,, \tag{2.9}$$

[17] One can also show that $\epsilon(g_i, g_i) = 1$ and $\epsilon(g_i, g_j)\epsilon(g_j, g_i) = 1$.

[18] The map $c : \overbrace{\Gamma \times \Gamma \ldots \times \Gamma}^{n} \longrightarrow \mathrm{U}(1)$ is an $n$-cochain. The set of all $n$-cochains forms an abelian group $C^n(\Gamma, \mathrm{U}(1))$ under multiplication. One can construct a coboundary operator $d_{n+1} : C^n(\Gamma, \mathrm{U}(1)) \longrightarrow C^{n+1}(\Gamma, \mathrm{U}(1))$ such that $d_n \circ d_{n+1} = 0$ and write down a corresponding complex. One can also define the group $Z^n(\Gamma, \mathrm{U}(1)) = \mathrm{Ker} d_{n+1}$ of $n$-cocycles and $B^n(\Gamma, \mathrm{U}(1)) = \mathrm{Im} d_n$ of $n$-coboundaries. The $n$-th cohomology group is defined as usual as $H^n(\Gamma, \mathrm{U}(1)) = \mathrm{Ker} d_{n+1}/\mathrm{Im} d_n$. For $n = 2$, a 2-cochain satisfying (2.5) maps to the identity under $d_3$, so it is a 2-cocycle. Moreover, $d_2 \circ c(g_i, g_j) = c_i c_j / c_{ij}$ so the equivalence classes of cocycles with (2.6) as an equivalence relation is given by $H^2(\Gamma, \mathrm{U}(1))$. Cocycles in the same conjugacy class are therefore cohomologous.

where $c \in \mathrm{U}(1)$. Associativity of matrix multiplication forces $c$ to satisfy the cocycle condition (2.5). Moreover, if $c$ satisfies (2.5), so does $c'$ defined in (2.6). The corresponding representation is trivially found to be $\gamma'(g_i) = c_i \gamma(g_i)$. Therefore, the different classes of projective representations of $\Gamma$ are also measured by $H^2(\Gamma, \mathrm{U}(1))$. Moreover, the invariant open string spectrum (2.7) of the orbifold model only depends on the cohomology class of the cocycle and not on the particular representative one chooses. This is complete analogy with the closed string discussion indicating that projective representations should be used when describing orbifolds with discrete torsion.

We will now show that once we make a particular choice of discrete torsion $\epsilon$ in (2.3) for the closed strings, that the action on the open string Chan-Paton factors is uniquely determined to be a projective representation (2.9) with cocycle $c$. This follows from a worldsheet CFT condition demanding $\Gamma$ to be a symmetry of the OPE. The action of $\Gamma$ on open and closed strings is consistent only if $\Gamma$ is conserved by interactions. We already know that this is the case for interactions involving only closed strings. One must also demand consistency of open-closed string interactions, that is $\Gamma$ has to be conserved by a open-closed string amplitude.[19] Let us consider for concreteness the transition between a Ramond-Ramond closed string state in the $g_i$-th twisted sector and photon arising from the open string ending on a D-brane transverse to the orbifold. To lowest order in the string coupling this amplitude arises in the disk. The closed string vertex operator is built out of a twist field which creates a cut from its location inside the disk to the boundary of the disk. Fields jump across the cut by the orbifold action $g_i$, which includes the action of $\gamma(g_i)$ on the Chan-Paton matrix $\lambda$ of the open string gauge field. This amplitude is completely determined by Lorentz invariance

$$\mathrm{tr}(\gamma(g_i)\lambda)\langle V^i_\alpha(0)\widetilde{V^i_\beta(0)}V^\mu(1)\rangle\,, \tag{2.10}$$

where $V^i_\alpha$, $\widetilde{V^i_\beta}$ are the right and left moving parts of the $g_i$-th twisted Ramond-Ramond vertex operator and $V^\mu$ is the vertex operator for the photon. Consistency requires invariance of this amplitude under the action of $\Gamma$. As mentioned earlier, the model is not specified until we choose a particular discrete torsion $\epsilon$ on the closed string. Therefore, taking into account how $\Gamma$ acts on closed string states (2.3) and the usual adjoint action (2.7) on open strings Chan-Paton factors, the amplitude (2.10) transforms under the action of $g_j$ as

$$\mathrm{tr}(\gamma(g_i)\gamma(g_j)^{-1}\lambda\gamma(g_j))\epsilon(g_i,g_j)\langle V^i_\alpha(0)\widetilde{V^i_\beta(0)}V^\mu(1)\rangle\,. \tag{2.11}$$

Invariance under $\Gamma$ requires setting equal (2.10) and (2.11), which gives after writing the discrete torsion phase in terms of cocycles the following constraint

$$\gamma(g_i)\gamma(g_j)c(g_j,g_i) = c(g_i,g_j)\gamma(g_j)\gamma(g_i)\,. \tag{2.12}$$

[19]A similar restriction was imposed by Polchinski [32] in orientifold models.

This constraint is satisfied by choosing a projective representation (2.9). Summarizing, we have shown from simple worldsheet principles that given an orbifold with discrete torsion (2.4) that the action of the orbifold group on open strings is determined by the corresponding cocycle.[20]

It is interesting to note that constraints on $\epsilon$ arise from a two-loop effect on closed strings but that consistent open-closed string interactions at tree level determine the action on the open strings.

## 3. Examples and D-brane charges

In this section we will work out a general class of examples and develop some of the relevant properties of projective representations of discrete groups that are needed to show the results anticipated in section 1 and 2. As mentioned in section 2 a discrete group $\Gamma$ admits projective representations if $H^2(\Gamma, \mathrm{U}(1))$ is non-trivial. The simplest abelian group admitting non-trivial projective representations — or equivalently, giving rise to discrete torsion in orbifolds — is $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_{n'}$. The allowed classes of representations are labeled by $H^2(\Gamma, \mathrm{U}(1)) \simeq \mathbb{Z}_d$, where $d = \gcd(n, n')$ is the greatest common divisor of $n$ and $n'$. Thus, a priori, there are $d$ different orbifold models one can define.

A basic definition in the theory of projective representations is that of a $c$-regular element. The number of irreducible projective representations of $\Gamma$ with cocycle $c$ equals the number of $c$-regular elements of $\Gamma$ [30].[21] A group element $g_i \in \Gamma$ — for abelian $\Gamma$ — is $c$-regular if

$$c(g_i, g_j) = c(g_j, g_i) \qquad \forall g_j \in \Gamma\,. \tag{3.1}$$

This definition is independent of the representative of the cocycle class. Thus, the number $N_c$ of irreducible projective representations with cocycle class $c$ is given by the following formula[22]

$$N_c = \frac{1}{|\Gamma|} \sum_{g_i, g_j \in \Gamma} \frac{c(g_i, g_j)}{c(g_j, g_i)} = \frac{1}{|\Gamma|} \sum_{g_i, g_j \in \Gamma} \epsilon(g_i, g_j)\,. \tag{3.2}$$

We have used (2.4) to write the above formula in terms of the discrete torsion phases. It is clear then that the closed string spectrum for the sectors twisted by $c$-regular

[20] There seems to be some arbitrariness in the projective representation one chooses. The worldsheet consistency condition only determines the cohomology class of the cocycle but does not pick a particular representative. This freedom, however, does not affect the spectrum of the orbifold model.

[21] This is very different to the case of vector representations, for which there are as many irreducible representations as there are conjugacy classes in the discrete group.

[22] We use the fact that any non-trivial one-dimensional representation yields zero when one sums over all group elements.

elements is the same as for conventional orbifolds, so that we have as many irreducible representations as massless Ramond-Ramond fields of a given rank. As explained in the section 1, this prediction should follow from the algebraic K-theory group $K_\Gamma^{[c]}(X)$ of twisted cross products.

Let's consider in some detail the example $\mathbb{Z}_n \times \mathbb{Z}_{n'}$. The discrete torsion phases appearing in the closed string partition function correspond to one dimensional representations of $\mathbb{Z}_n \times \mathbb{Z}_{n'}$. If we let $g_1$ be the generator of $\mathbb{Z}_n$ and $g_2$ the generator of $\mathbb{Z}_{n'}$ a general group element can be written as $g_1^a g_2^b$, where $a$ and $b$ are integers. Then, the allowed discrete torsion phases are

$$\epsilon(ab, a'b') = \alpha^{m(ab'-a'b)}\,, \qquad m = 0, \ldots, d-1\,, \tag{3.3}$$

where $\alpha = \exp(2\pi i/d)$ and $d = \gcd(n, n')$. As expected there are $d$ different phases one can associate to the closed string partition function (2.1).

The study of D-branes in these backgrounds require analyzing the representation theory[23] of $\mathbb{Z}_n \times \mathbb{Z}_{n'}$. For our purposes, we only need to find the number of irreducible representations in a given cohomology class and their dimensionality. We can use (3.2) and (3.3) to find how many of them there are. Let $p$ be the smallest non-zero integer such that

$$\exp\left(\frac{2\pi imp}{d}\right) = 1\,, \tag{3.4}$$

then the sum (3.2) can be split into sums of blocks of $p$ elements. Usual vector representations correspond to $p = 1$. If we perform the sum over say $a$ and $b$ for each block we get 0 except when $a'$ and $b'$ are multiples of $p$, for which the sum over $a$ and $b$ over a block of $p$-elements just gives $p^2$. Since there are $n/p$ and $n'/p$ blocks of $p$ elements for the sum over $a$ and $b$ and $a'$ and $b'$ respectively, the total sum yields

$$N_c = \frac{1}{nn'}\left(\frac{nn'}{p^2}\right)^2 p^2 = \frac{nn'}{p^2}\,. \tag{3.5}$$

Therefore, there are $N_c = nn'/p^2$ irreducible projective representations with cocycle $c$ for $\mathbb{Z}_n \times \mathbb{Z}_{n'}$. The dimensionality of each irreducible representation can be obtained from the fact that the regular representation can be decomposed in terms irreducible ones

$$|\Gamma| = \sum_{a=1}^{N_c} d_{R_a}^2\,, \tag{3.6}$$

where $R_a$ labels the different irreducible representations. Thus, each representation is $p$-dimensional. Usual vector representations ($c \equiv 1$) are one-dimensional and all irreducible projective one are bigger.

[23]This example has also been considered recently by [19].

The conclusion stating that the minimal charge of a D-brane configuration wrapping the entire compact orbifold is an integer bigger than the minimal charge whenever discrete torsion is non-trivial can be verified by a simple disk amplitude. We want to compute the charge under the untwisted sector Ramond-Ramond field corresponding to a wrapped D6-brane. This can be computed by inserting the untwisted six-brane Ramond-Ramond vertex operator on the disk. We will sketch the computation and refer to [26] for more details. The vertex operator has to be in the $(-3/2, -1/2)$ picture to soak the background superghost charge on the disk. In this picture and the Ramond-Ramond potential $C_{\mu_0 \ldots \mu_6}$ appears in the vertex operator. The amplitude is multiplied by the trace of the representation acting on Chan-Paton factors for the identity element (to compute the charge under the $g_i$ Ramond-Ramond field one multiplies by the trace of the representation for $g_i$). The computation can be easily computed by conformally mapping onto the upper half plane and imposing the appropriate boundary conditions. The final result is [26]

$$Q = \frac{d_R}{|\Gamma|}, \tag{3.7}$$

where $d_R$ is the dimension of the representation considered. This formula applies both for conventional orbifolds as well as for orbifolds with discrete torsion. In the first case one must use projective representations and in the second vector representations. Since the smallest irreducible vector representations of $\Gamma$ is one-dimensional but the smallest irreducible projective representation is larger, this shows that indeed the minimal D-brane charge allowed for orbifolds with discrete torsion are bigger than for conventional orbifolds.

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